Foundations of fractional calculus

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Calculus

Modern calculus was founded in the 17th century by Isaac Newton and Gottfried Leibniz.

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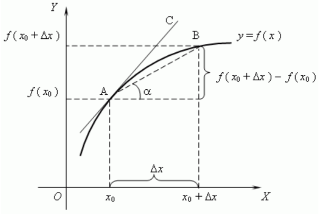
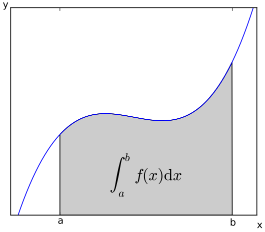
- derivatives

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

- integrals

$$F(x) = \int f(x)\, dx \;\Rightarrow\; F'(x) = f(x)$$





Fractional Calculus was born in 1695

$$\frac{d^n f}{dt^n}$$

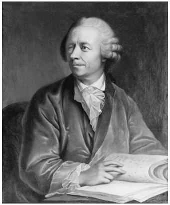
What if the order will be $n = \frac{1}{2}$?

It will lead to a paradox, from which one day useful consequences will be drawn.

G.F.A. de L'Hôpital
(1661–1704)

G.W. Leibniz
(1646–1716)

Leonhard Euler (1707-1783)



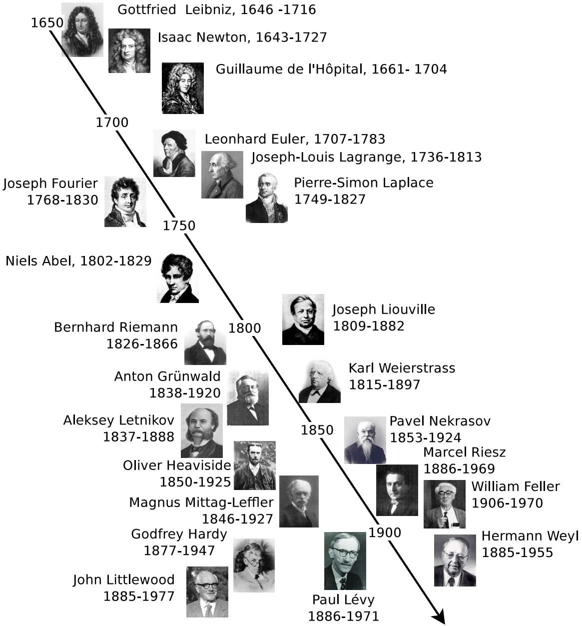
in 1738 gave a meaning to the derivative $\dfrac{d^\alpha(x^n)}{dx^\alpha}$, for $\alpha \notin \mathbb{N}$

Joseph Liouville (1809-1882)

in his papers in 1832-1837 gave a solid foundation to the fractional calculus, which has undergone only slight changes since then.

Georg Friedrich Bernhard Riemann (1826-1866)

in a paper from 1847 which was published 29 years later (and ten years after his death) proposed a definition for fractional integration in the form that is still in use today.


1650
Gottfried Leibniz, 1646 -1716
Isaac Newton, 1643-1727
Guillaume de l'Hôpital, 1661- 1704
1700
Leonhard Euler, 1707-1783
Joseph-Louis Lagrange, 1736-1813
Joseph Fourier
1768-1830
Pierre-Simon Laplace
1749-1827
1750
Niels Abel, 1802-1829
Joseph Liouville
1809-1882
Bernhard Riemann
1826-1866
1800
Anton Grünwald
1838-1920
Karl Weierstrass
1815-1897
Aleksey Letnikov
1837-1888
1850
Pavel Nekrasov
1853-1924
Marcel Riesz
1886-1969
Oliver Heaviside
1850-1925
William Feller
1906-1970
Magnus Mittag-Leffler
1846-1927
Godfrey Hardy
1877-1947
1900
Hermann Weyl
1885-1955
John Littlewood
1885-1977
Paul Lévy
1886-1971

Special functions

Special functions, such as the gamma function, the beta function or the Mittag-Leffler function, have an important role in fractional calculus.

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- The gamma function:

$$\Gamma(z) := \int_0^\infty e^{-t} t^{z-1}\, dt \qquad (\operatorname{Re} z > 0)$$

with properties

$$\Gamma(z+1) = z\,\Gamma(z), \qquad \Gamma(n+1) = n! \;\; (n \in \mathbb{N}).$$

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- The beta function:

$$B(z,\omega) = \int_0^1 t^{z-1}(1-t)^{\omega-1}\, dt \qquad (\operatorname{Re} z, \operatorname{Re}\omega > 0)$$

with properties

$$B(z,\omega) = \frac{\Gamma(z)\Gamma(\omega)}{\Gamma(z+\omega)}.$$

- The Mittag-Leffler function:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha\, k + \beta)} \qquad (\operatorname{Re}\alpha, \operatorname{Re}\beta > 0)$$

  with special cases:

  - $E_{0,1}(z) = \dfrac{1}{1-z}$
  - $E_{1,1}(z) = e^z$
  - $E_{2,1}(z) = \cosh(\sqrt{z})$

The Riemann-Liouville fractional calculus

Let $u \in L^1([a,b])$ and $\alpha > 0$.

- The left Riemann-Liouville fractional integral of order α:

$$ {}_aI_t^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\theta)^{\alpha-1} u(\theta)\, d\theta. $$

- The right Riemann-Liouville fractional integral of order α:

$$ {}_tI_b^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\theta-t)^{\alpha-1} u(\theta)\, d\theta. $$

Properties:

- If $\alpha = n \in \mathbb{N}$ then

$$ {}_aI_t^n u(t) = \frac{1}{(n-1)!}\int_a^t (t-\theta)^{n-1} u(\theta)\, d\theta = \int_a^{t_n}\int_a^{t_{n-1}}\dots\int_a^{t_1} u(\theta)\, d\theta_1 \dots d\theta_n, $$

  i.e., ${}_aI_t^n u$ is just an n-fold integral of u.
- Semigroup property:

$$ {}_aI_t^\alpha\, {}_aI_t^\beta u = {}_aI_t^{\alpha+\beta} u \quad \text{and} \quad {}_tI_b^\alpha\, {}_tI_b^\beta u = {}_tI_b^{\alpha+\beta} u \quad (\alpha, \beta > 0) $$

  Consequences:
  - ${}_aI_t^\alpha$ and ${}_aI_t^\beta$ commute, i.e., ${}_aI_t^\alpha\, {}_aI_t^\beta = {}_aI_t^\beta\, {}_aI_t^\alpha$;
  - ${}_aI_t^\alpha (t-a)^\mu = \dfrac{\Gamma(\mu+1)}{\Gamma(\mu+1+\alpha)} \cdot (t-a)^{\mu+\alpha}$.

Let $u \in AC([a,b])$ and $0 \leq \alpha < 1$.

- The left Riemann-Liouville fractional derivative of order α:

$$ {}_aD_t^{\alpha} u(t) = \frac{d}{dt} {}_aI_t^{1-\alpha} u(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{u(\theta)}{(t-\theta)^{\alpha}} \, d\theta. $$

- The right Riemann-Liouville fractional derivative of orderα:

$$ {}_tD_b^{\alpha} u(t) = \Big(- \frac{d}{dt} \Big) {}_tI_b^{1-\alpha} u(t) = \frac{1}{\Gamma(1-\alpha)} \Big(- \frac{d}{dt} \Big) \int_t^b \frac{u(\theta)}{(\theta - t)^{\alpha}} \, d\theta. $$

- If $\alpha = 0$ then ${}_aD_t^0 u(t) = {}_tD_b^0 u(t) = u(t)$.

Euler formula:

$$ {}_aD_t^\alpha (t-a)^{-\mu} = \frac{\Gamma(1-\mu)}{\Gamma(1-\mu-\alpha)} \frac{1}{(t-a)^{\mu+\alpha}} $$

Derivation:

$$ \begin{aligned} {}_aD_t^\alpha (t-a)^{-\mu} &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{(\theta-a)^{-\mu}}{(t-\theta)^\alpha}\, d\theta \\ &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^{t-a} \frac{z^{-\mu}}{(t-a-z)^\alpha}\, dz \\ &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^{t-a} \frac{z^{-\mu}}{(t-a)^\alpha (1-\frac{z}{t-a})^\alpha}\, dz \\ &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^1 \frac{(t-a)^{-\mu}\xi^{-\mu}}{(t-a)^\alpha(1-\xi)^\alpha}\, (t-a) d\xi \\ &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} (t-a)^{1-\mu-\alpha} \int_0^1 \xi^{-\mu}(1-\xi)^{-\alpha}\, d\xi \\ &= \frac{1-\mu-\alpha}{\Gamma(1-\alpha)} \cdot (t-a)^{-\mu-\alpha} \cdot B(1-\mu, 1-\alpha) \\ &= \frac{\Gamma(1-\mu)}{\Gamma(1-\mu-\alpha)} \cdot (t-a)^{-\mu-\alpha} \end{aligned} $$

Conclusions:

- $\mu = 1-\alpha$: ${}_aD_t^\alpha (t-a)^{\alpha-1} = 0$
- $\mu = 0$: ${}_aD_t^\alpha c \neq 0$, for any constant $c \in \mathbb{R}$.

Definition of R-L fractional derivatives can be extended for any $\alpha \geq 1$: write $\alpha = [\alpha] + \{\alpha\}$, where $[\alpha]$ denotes the integer part and $\{\alpha\}$, $0 \leq \{\alpha\} < 1$, the fractional part of α.
Let $u \in AC^n([a,b])$ and $n-1 \leq \alpha < n$.

- The left Riemann-Liouville fractional derivative of order α:

$$ {}_aD_t^\alpha u(t) = \left(\frac{d}{dt}\right)^{[\alpha]} {}_aD_t^{\{\alpha\}} u(t) = \left(\frac{d}{dt}\right)^{[\alpha]+1} {}_aI_t^{1-\{\alpha\}} u(t) $$

  or

$$ {}_aD_t^\alpha u(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t \frac{u(\theta)}{(t-\theta)^{\alpha-n+1}}\, d\theta $$

- The right Riemann-Liouville fractional derivative of order α:

$$ {}_tD_b^\alpha u(t) = \left(-\frac{d}{dt}\right)^{[\alpha]} {}_tD_b^{\{\alpha\}} u(t) = \left(-\frac{d}{dt}\right)^{[\alpha]+1} {}_tI_b^{1-\{\alpha\}} u(t) $$

  or

$$ {}_tD_b^\alpha u(t) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dt}\right)^n \int_t^b \frac{u(\theta)}{(\theta-t)^{\alpha-n+1}}\, d\theta $$

Properties:

- ${}_aD_t^\alpha\, {}_aI_t^\alpha u = u$
- ${}_aI_t^\alpha\, {}_aD_t^\alpha u \neq u$, but

$$ {}_aI_t^\alpha\, {}_aD_t^\alpha u(t) = u(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{\alpha-k-1}}{\Gamma(\alpha-k)} \frac{d^{n-k-1}}{dt}\bigg|_{t=a} \Big({}_aI_t^{n-\alpha} u(t)\Big) $$

- Contrary to fractional integration, Riemann-Liouville fractional derivatives do not obey either the semigroup property or the commutative law: E.g. $u(t) = t^{1/2}, a = 0, \alpha = 1/2$ and $\beta = 3/2$.

$$ {}_0D_t^\alpha u = \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi}, \quad {}_0D_t^\beta u = 0 $$

$$ {}_0D_t^\alpha\Big({}_0D_t^\beta u\Big) = 0, \quad {}_0D_t^\beta\Big({}_0D_t^\alpha u\Big) = -\frac{1}{4}\, t^{-\frac{3}{2}}, \quad {}_0D_t^{\alpha+\beta} u = -\frac{1}{4}\, t^{-\frac{3}{2}} $$

$$ \Rightarrow {}_0D_t^\alpha({}_0D_t^\beta u) \neq {}_0D_t^{\alpha+\beta} u \quad \text{and} \quad {}_0D_t^\alpha({}_0D_t^\beta u) \neq {}_0D_t^\beta({}_0D_t^\alpha u) $$

Left Riemann-Liouville fractional operators via convolutions

Set

$$f_\alpha(t) := \begin{cases} H(t)\dfrac{t^{\alpha-1}}{\Gamma(\alpha)}, & \alpha > 0 \\ \left(\dfrac{d}{dt}\right)^N f_{\alpha+N}(t), & \alpha \le 0, N \in \mathbb{N} : N+\alpha > 0 \wedge N+\alpha-1 \le 0 \end{cases}$$

(H is the Heaviside function)

- $\alpha \in \mathbb{N}$:

$$f_1(t) = H(t), \quad f_2(t) = t_+, \quad f_3(t) = \frac{t_+^2}{2}, \quad \dots \quad f_n(t) = \frac{t_+^{n-1}}{(n-1)!},$$

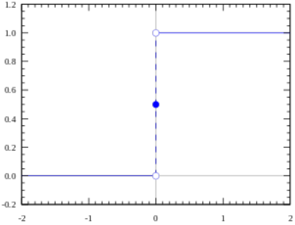
- $-\alpha \in \mathbb{N}_0$:

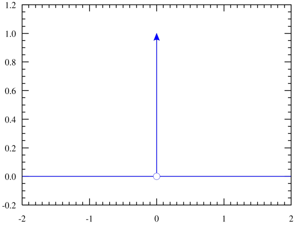
$$f_0(t) = \delta(t), \quad f_{-1}(t) = \delta'(t), \quad \dots \quad f_{-n}(t) = \delta^{(n)}(t).$$

- The convolution operator $f_\alpha *$ is the left Riemann-Liouville operator of
    - differentiation for $\alpha < 0$
    - integration for $\alpha \ge 0$

Heaviside function:

$$H(t) = \begin{cases} 0, & t < 0 \\ \frac{1}{2}, & t = 0 \\ 1 & t > 0 \end{cases}$$





Dirac delta distribution

generalized function that vanishes everywhere except at the origin, with integral 1 over $\mathbb{R}$

Convolution

$$f * u(t) = \int_{-\infty}^{+\infty} f(t-\theta)u(\theta)\, d\theta$$

Analytical expressions of some fractional derivatives

$f(x), x > a$	${}^{\mathrm{RL}}D^{\alpha}_{a+} f(x)$
k	$\frac{k(x-a)^{-\alpha}}{\Gamma(1-\alpha)}$
$(x-a)^{\beta},\ \Re(\beta) > -1$	$\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}\,(x-a)^{\beta-\alpha}$
$e^{\lambda x},\ \lambda \neq 0$	$e^{\lambda a}(x-a)^{-\alpha} E_{1,1-\alpha}(\lambda(x-a)) =$ $e^{\lambda a}\mathscr{E}_{x-a}(-\alpha,\lambda)$
$(x \pm p)^{\lambda},\ a \pm p > 0$	$\frac{(a\pm p)^{\lambda}}{\Gamma(1-\alpha)}\,(x-a)^{-\alpha}{}_2F_1\left(1,-\lambda,1-\alpha;\frac{a-x}{a\pm p}\right)$
$(x-a)^{\beta}(x\pm p)^{\lambda}$, $\Re(\beta) > -1 \wedge\ a \pm p > 0$	$\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}\,(a\pm p)^{\lambda}(x-a)^{\beta-\alpha}\cdot$ $\cdot{}_2F_1\left(\beta+1,-\lambda;\beta-\alpha;\frac{a-x}{a\pm p}\right)$
$(x-a)^{\beta}(p-x)^{\lambda}$, $p > x > a \wedge \Re(\beta) > -1$	$\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}\,(p-a)^{\lambda}(x-a)^{\beta-\alpha}\cdot$ $\cdot{}_2F_1\left(\beta+1,-\lambda;\beta-\alpha;\frac{x-a}{p-a}\right)$
$(x-a)^{\beta}\,e^{\lambda x},\ \Re(\beta) > -1$	$\frac{\Gamma(\beta+1)e^{\lambda a}}{\Gamma(\beta+1-\alpha)}\,(x-a)^{\beta-\alpha}\cdot$ $\cdot{}_1F_1(\beta+1,\beta+1-\alpha;\lambda(x-a))$
$\sin(\lambda(x-a))$	$\frac{(x-a)^{-\alpha}}{2i\Gamma(1-\alpha)}\cdot[{}_1F_1(1,1-\alpha,i\lambda(x-a))-$ ${}_1F_1(1,1-\alpha,-i\lambda(x-a))]$

Caputo fractional derivatives

Let $u \in AC^n([a,b])$ and $n-1 \leq \alpha < n$.

- The left Caputo fractional derivative of order α:

$$ {}_a^c D_t^\alpha u(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{u^{(n)}(\theta)}{(t-\theta)^{\alpha-n+1}} \, d\theta $$

- The right Caputo fractional derivative of order α:

$$ {}_t^c D_b^\alpha u = \frac{1}{\Gamma(n-\alpha)} \int_t^b \frac{(-u)^{(n)}(\theta)}{(\theta-t)^{\alpha-n+1}} \, d\theta $$

Properties:

- In general, R-L and Caputo fractional derivatives do not coincide:

$$ {}_aD_t^\alpha u = \left(\frac{d}{dt}\right)^{[\alpha]} {}_aI_t^{1-\{\alpha\}} u \neq {}_aI_t^{1-\{\alpha\}} \left(\frac{d}{dt}\right)^{[\alpha]} u = {}_a^cD_t^\alpha u, $$

  but

$$ {}_aD_t^\alpha u = {}_a^cD_t^\alpha u + \sum_{k=0}^{n-1} \frac{(t-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} u^{(k)}(a^+), $$

  and

$$ {}_tD_b^\alpha u = {}_t^cD_b^\alpha u + \sum_{k=0}^{n-1} \frac{(b-t)^{k-\alpha}}{\Gamma(k-\alpha+1)} u^{(k)}(b^-). $$

- If $u^{(k)}(a^+) = 0$ $(k = 0, 1, \ldots, n-1)$ then ${}_aD_t^\alpha u = {}_a^cD_t^\alpha u$;
  If $u^{(k)}(b^-) = 0$ $(k = 0, 1, \ldots, n-1)$ then ${}_tD_b^\alpha u = {}_t^cD_b^\alpha u$.
- ${}_a^cD_t^\alpha k = {}_t^cD_b^\alpha k = 0$, $k \in \mathbb{R}$.

Fractional identities

- Linearity:

$$ {}_aD_t^\alpha(\mu u + \nu v) = \mu \cdot {}_aD_t^\alpha u + \nu \cdot {}_aD_t^\alpha v $$

- The Leibnitz rule does not hold in general:

$$ {}_aD_t^\alpha(u \cdot v) \neq u \cdot {}_aD_t^\alpha v + {}_aD_t^\alpha u \cdot v $$

- For analytic functions u and v:

$$ {}_aD_t^\alpha(u \cdot v) = \sum_{i=0}^{\infty} \binom{\alpha}{i} ({}_aD_t^{\alpha - i} u) \cdot v^{(i)}, \qquad \binom{\alpha}{i} = \frac{(-1)^{i-1}\alpha\Gamma(i-\alpha)}{\Gamma(1-\alpha)\Gamma(i+1)} $$

- For a real analytic function u:

$$ {}_aD_t^\alpha u = \sum_{i=0}^{\infty} \binom{\alpha}{i} \frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)} u^{(i)}(t) $$

- Fractional integration by parts:

$$ \int_a^b f(t)\, {}_aD_t^\alpha g \, dt = \int_a^b g(t)\, {}_tD_b^\alpha f \, dt $$

Laplace and Fourier transform

- Fourier transform:

$$\mathcal{F}u(\omega) = \hat{u}(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega x} u(x)\, dx \qquad (\omega \in \mathbb{R})$$

- Laplace transform:

$$\mathcal{L}u(s) = \tilde{u}(s) = \int_0^{+\infty} e^{-st} u(t)\, dt \qquad (\mathrm{Re}\, s > 0)$$

Properties:

$$\mathcal{F}[D^n u](\omega) = (i\omega)^n \mathcal{F}u(\omega); \quad \mathcal{L}[D^n u](s) = s^n \mathcal{L}u(s) - \sum_{k=0}^{n-1} s^{n-k-1} D^k u(a)$$

$$\mathcal{F}[D^\alpha u](\omega) = (i\omega)^\alpha \mathcal{F}u(\omega); \quad \mathcal{L}[D^\alpha u](s) = s^\alpha \mathcal{L}u(s) - \sum_{k=0}^{n-1} s^{n-k-1} D^k I^{n-\alpha} u(a)$$

$\mathscr{L}\{f(t)\}(s)$	$f(t)$
$\frac{k!s^{\alpha-\beta}}{(s^\alpha \mp a)^{k+1}}$	$t^{\alpha k+\beta-1}\frac{\mathrm{d}^k E_{\alpha,\beta}(\pm a t^\alpha)}{\mathrm{d}(\pm a t^\alpha)^k}$
$\frac{1}{s^\alpha-\lambda}$	$e_\alpha^{\lambda t}$
$\frac{n!s^{\alpha-1}}{(s^\alpha-\lambda)^{n+1}}$	$t^{\alpha n}\left(\frac{\partial}{\partial\lambda}\right)^n E_\alpha(\lambda t^\alpha)$
$\frac{n!}{(s^\alpha-\lambda)^{n+1}}$	$\left(\frac{\partial}{\partial\lambda}\right)^n e_\alpha^{\lambda z}$
$\frac{s^{\alpha-\beta}}{s^\alpha \mp a}$	$t^{\beta-1}E_{\alpha,\beta}(\pm a t^\alpha)$
$\frac{s^{\alpha-1}}{s^\alpha \mp a}$	$E_\alpha(\pm a t^\alpha)$
$\frac{1}{s^\alpha \mp a}$	$t^{\alpha-1}E_{\alpha,\alpha}(\pm a t^\alpha)$
$\frac{s^{1-\beta}}{s \mp a}$	$t^{\beta-1}E_{1,\beta}(\pm a t) = \mathscr{E}_t(\beta-1,\pm a)$
$\frac{1}{s^\beta}$	$t^{\beta-1}E_{1,\beta}(0) = \mathscr{E}_t(\beta-1,0) = \frac{t^{\beta-1}}{\Gamma(\beta)}$
$\frac{1}{\sqrt{s}}$	$\frac{1}{\sqrt{\pi t}}$
$\frac{1}{s\sqrt{s}}$	$2\sqrt{\frac{t}{\pi}}$
$\frac{1}{s^n\sqrt{s}}$, $(n=1,2,\cdots)$	$\frac{2^n t^{n-\frac{1}{2}}}{1\cdot 3\cdot 5\cdots(2n-1)\sqrt{\pi}}$
$\frac{s}{(s-a)^{\frac{3}{2}}}$	$\frac{1}{\sqrt{\pi t}}e^{at}(1+2at)$
$\sqrt{s-a}-\sqrt{s-b}$	$\frac{1}{2\sqrt{\pi t^3}}\left(e^{bt}-e^{at}\right)$
$\frac{1}{\sqrt{s}+a}$	$\frac{1}{\sqrt{\pi t}} - a e^{a^2 t}\operatorname{erfc}\left(a\sqrt{t}\right)$
$\frac{\sqrt{s}}{s-a^2}$	$\frac{1}{\sqrt{\pi t}} + a e^{a^2 t}\operatorname{erf}\left(a\sqrt{t}\right)$
$\frac{\sqrt{s}}{s+a^2}$	$\frac{1}{\sqrt{\pi t}} - \frac{2a}{\sqrt{\pi}}e^{-a^2 t}\int_0^{a\sqrt{t}} e^{\tau^2}\mathrm{d}\tau$
$\frac{1}{\sqrt{s}(s-a^2)}$	$\frac{1}{a}e^{a^2 t}\operatorname{erf}\left(a\sqrt{t}\right)$
$\frac{1}{\sqrt{s}(s+a^2)}$	$\frac{2}{a\sqrt{\pi}}e^{-a^2 t}\int_0^{a\sqrt{t}} e^{\tau^2}\mathrm{d}\tau$

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